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The mesofractal model of the large scale structure of the universe

The report is devoted to description of the observed non-homogeneous structure of the Universe in terms of the walk model with a transition probability of the inverse power type called the Rayleigh-Lèvy (RL)walk. This model contains four free parameters. An appropriate choice of them yields the mesofractal structure, revealing fractal properties on small scales and looking like a homogeneous medium on large scales. This approach can be considered as some kind of descriptive statistics allowing to extract necessary characteristics of the structure using a few parameters.

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1. Introduction

One of important problems connected to the large scale structure of the Universe is searching the most efficient ways of contraction of the information obtained from observations. Historically, the first way consisted in description of isolated clumps, clumps of clumps and so on by means of some parameters. Nowadays, statistical approaches involving large numbers of such objects play the leading role in investigation of the problem. One can point at the cell method [1], the distance distribution method [2], the correlation method [3], [4], the power spectrum analysis [5], the topological analysis [6], the wavelet analysis [7], the percolation analysis [8], the scaling analysis [9], the fractal and multifractal analysis [10] and others.

Angular galaxy samples, three-dimensional samples, cluster samples, galaxy redshift surveys are the least contracted information. The most contracted information is represented by only one number ρ being the mass density of the Universe. Note, that any way of contracting requires some apriori assumptions. In the first case, the assumptions are put into mathematical algorithms used for processing of observations. In the second example, the additional but very important assumptions are used: (i) the Cosmological Principle: *There are no exist of privileged points in the Universe* [11], (ii) the Ergodic Principle: *Ensembles averages equal spatial averages taken over one realization of the random field*, and (iii) the assumption about *homogeneoty and isotropy on large scale*. Peebles [3] combines them and calls the resulting set of the assumptions the *fair sample hypothesis*.

To investigate the relation between the principles and the observations data, some statistical models have been offered. One of them, offered by Mandelbrot [12], [13] is based on a random walk process. He had drown attention to the fact, that random sets of nodes generated by the Rayleigh-Lèvy (RL) walk reveal long-range correlations of inverse power type similar to observed galaxy correlations, and introduced concept of the *stochastic fractal*.

The stochastic fractal is determined as a self-similar (in the stochastic sense) structure with the same probabilistic property on all scales [13]. Applied to the galaxy distribution it means the presence of the over-dense regions and voids extending to all scales. Namely these properties are inherent in RL trajectories. Nevertheless, a majority of cosmologists believe that the Universe is not self-similar structure and looks like homogeneous medium on large scales. This point of view demands some modifying the model. P.Coleman and L.Pietronero write:"A sample which is fractal on small scales and homogeneous on large scales is constructed as follows: A number of random locations are chosen in a large volume. Since the locations are Poissonly distributed, they will have an average separation $\lambda_0 \dots$ Each of these locations is the starting point for constructing a fractal which samples length scales up to the limiting value λ_0" [10].We call the system *mesofractal* to avoid the more cumbersome term "fractal with a turnover to homogeneity" used in the cited work.

The aim of this work is to investigate the mesofractal model of the large-scale structure of the Universe in more detail. By the reasons explained in [3], we will ignore the curvature of the space-time and the expansion of the Universe.

2. Correlation concept

Let $\mathbf{X}_1, \mathbf{X}_2, \dots$ be a set of points representing galaxy positions in 3-dimensional Euclidian space and $N(V)$ be a number of the galaxies in the domain V. As a function of V, this number is nothing but an integer-valued measure, because it is non-negative and σ-additive function of sets being equal

to zero for empty domain V (we will denote the domain and its volume by means of the same letter V).

Following Gibbs' ideas, one may consider this set as a particular realization of some statistical ensemble of galaxy set in space governed by some random measure $N(V)$. Using $\langle \dots \rangle$ as the symbol of averaging over the random measure, we write the expression of the generating functional (GF) of the random distribution $\mathbf{X}_j$

$$F(u(\cdot)) = \langle \prod_j u(\mathbf{X}_j) \rangle = \langle \exp \int \ln u(\mathbf{x}) N(d\mathbf{x}) \rangle,$$

where $0 < u(\mathbf{x}) \le 1$ is the argument of the functional. This value considered as a functional of $u(\mathbf{x})$ is an exhaustive characteristics of the random measure $N(V)$. Two kinds of densities can be derived from it by means of the functional differentiation:

$$p_k(\mathbf{x}_1, \dots, \mathbf{x}_k) = \left. \frac{\delta^k F(u(\cdot))}{\delta u(\mathbf{x}_1) \dots \delta u(\mathbf{x}_k)} \right|_{u=0}$$

and

$$f_k(\mathbf{x}_1, \dots, \mathbf{x}_k) = \left. \frac{\delta^k F(u(\cdot))}{\delta u(\mathbf{x}_1) \dots \delta u(\mathbf{x}_k)} \right|_{u=1}.$$

The following relations take place:

$$\frac{1}{k!} \int\limits_V \dots \int\limits_V p_k(\mathbf{x}_1, ..., \mathbf{x}_k) d\mathbf{x}_1 ... d\mathbf{x}_k = \mathrm{Prob}\{N(V) = k\}$$

and

$$\int\limits_V \dots \int\limits_V f_k(\mathbf{x}_1, ..., \mathbf{x}_k) d\mathbf{x}_1 ... d\mathbf{x}_k = \langle N^{[k]}(V) \rangle,$$

where $\langle N^{[k]}(V) \rangle \equiv \langle N(N-1)...(N-k+1) \rangle$ is the k-th factorial moment of the random, measure $N(V)$, so that $f_k(\mathbf{x}_1, ..., \mathbf{x}_k)$ is the *factorial moment density* (fmd) of the k-th order.

Any characteristics of the visible galaxy distribution in an observation domain V, $H(N(\cdot))$ can be given by a set of functions $h_0, h_1(\mathbf{x}_1), h_2(\mathbf{x}_1, \mathbf{x}_2), ...,$ symmetrical with respect to permutation of their arguments (if we ignore distinctions between galaxy masses). The expectation value of $H(N(\cdot))$ can be expressed through the probability densities:

$$\langle H(N(\cdot)) \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\mathbf{x}_1 ... \int d\mathbf{x}_n h_n(\mathbf{x}_1, ..., \mathbf{x}_n) p_n(\mathbf{x}_1, ..., \mathbf{x}_n)$$

and through the fmd's:

$$\langle H(N(\cdot)) \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\mathbf{x}_1 ... \int d\mathbf{x}_n h_n^+(\mathbf{x}_1, ..., \mathbf{x}_n) f_n(\mathbf{x}_1, ..., \mathbf{x}_n),$$

where

$$h_n^+(\mathbf{x}_1, ..., \mathbf{x}_n) = \sum_{k=0}^{n} \frac{(-)^{n-k}}{k!} \sum_{i_1 \neq \dots \neq i_k}^{n} h_k(\mathbf{x}_{i_1}, ..., \mathbf{x}_{i_n}).$$

Note, that p_0 stands for the probability that the domain V turns out to be empty, while $f_0 = 1$. Detailed discussion of the relations can be found in the books [14]- [16].

The simplest kind of statistical ensembles is the homogeneous Poisson ensemble formed by independent random points. It is characterized by GF

$$F_0(u(\cdot)) = \exp\left\{ \int [u(\mathbf{x}) - 1] f_1(\mathbf{x}) d\mathbf{x} \right\} = \exp\left\{ n_0 \int [u(\mathbf{x}) - 1] d\mathbf{x} \right\},$$

where $n_0 \equiv f_1(\mathbf{x})$ is the mean concentration of galaxies at point $\mathbf{x}$. As a result we obtain

$$f_k(\mathbf{x}_1, ..., \mathbf{x}_k) = f_1(\mathbf{x}_1) ... f_1(\mathbf{x}_k).$$

In a general case, when the random points are correlated, the GF can be represented in the form

$$F(u(\cdot)) = 1 + \sum_{r=1}^{\infty} \frac{(-)^k}{k!} \int d\mathbf{x}_1 \dots \int d\mathbf{x}_k \, f_k(\mathbf{x}_1, \dots, \mathbf{x}_k) [1 - u(\mathbf{x}_1)] \dots [1 - u(\mathbf{x}_k)].$$

The densities f_k of orders $k > 1$ are usually represented by means of *irreducible correlation functions* $\varphi_k(\mathbf{x}_1, \dots, \mathbf{x}_k)$ [15]:

$$f_2(\mathbf{x}_1, \mathbf{x}_2) = f_1(\mathbf{x}_1) f_1(\mathbf{x}_2) + \varphi_2(\mathbf{x}_1, \mathbf{x}_2),$$

$$f_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \stackrel{s}{=} f_1(\mathbf{x}_1) f_1(\mathbf{x}_2) f_1(\mathbf{x}_3) + 3 f_1(\mathbf{x}_1) \varphi_2(\mathbf{x}_2, \mathbf{x}_3) + \varphi_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$$

and so on, where $\stackrel{s}{=}$ means that the right side is symmetrized over all permutations of the arguments. Denoting the right side of the equality by c_{123} we present the operation in explicit form:

$$f_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \{c_{123} + c_{132} + c_{213} + c_{231} + c_{312} + c_{321}\}/3!.$$

The functions φ_k are equal to 0 for Poisson's ensembles but this is not a case for the real galaxy distribution. For its description is commonly used a little another set of functions [3,17]:

$$\xi_k(\mathbf{x}_1, ..., \mathbf{x}_k) = \varphi_k(\mathbf{x}_1, ..., \mathbf{x}_k)/[f_1(\mathbf{x}_1) \dots f_1(\mathbf{x}_k)].$$

The most important of them is the two-point correlation function $\xi_2(\mathbf{x}_1, \mathbf{x}_2)$. Its physical sense can be made clear via the relation

$$[1 + \xi_2(\mathbf{x}_1, \mathbf{x}_2)] f_1(\mathbf{x}_1) f_1(\mathbf{x}_2) d\mathbf{x}_1 d\mathbf{x}_2 \sim \langle N^{[2]}(d\mathbf{x}_1, d\mathbf{x}_2)\rangle \sim P^{(2)}(d\mathbf{x}_1, d\mathbf{x}_2),$$

where $\langle N^{[2]}(d\mathbf{x}_1, d\mathbf{x}_2)\rangle$ is the mean number of pairs of galaxies belonging to infinitesimal volumes $d\mathbf{x}_1$ and $d\mathbf{x}_2$ while $P^{(2)}(d\mathbf{x}_1, d\mathbf{x}_2)$ is the probability that $d\mathbf{x}_1$ contents some galaxy and $d\mathbf{x}_2$ contents some another galaxy. The function ξ is an accurate instrument for comparing theoretical models with observational data. It is used for calculation of the clumping energy and some other important characteristics.

3. Two kinds of statistics

The following conditions define the *fair sample*:

1. $f_1(\mathbf{x}) = n = \text{const};$

2. $\xi_2(\mathbf{x}_1, \mathbf{x}_2) = \xi(r_{12}), \quad r_{12} \equiv |\mathbf{x}_1 - \mathbf{x}_2|;$

3. $\xi(r) \to 0$ as $r \to 0$,

4. $\xi_k(\mathbf{x}_1, \dots, \mathbf{x}_k) = \xi_k(\mathbf{x}'_1, \dots, \mathbf{x}'_k)$

where $\mathbf{x}'_j$ are coordinates of the points $\mathbf{x}_j$ with respect to any translated or turned coordinate system.

The latter assumption claiming statistical equivalence of all *space points* with respect to their environment is nothing but statistical version of the Cosmological Principle. This is a base for statistics of the first kind - unconditional statistics. In its frame, it is of no importance if an origin of coordinates is occupied by some galaxy or not: the unconditional correlation functions ξ_k contain information averaged over the whole Gibbs' ensemble of the random medium. This approach is widely used in statistical mechanics [14,15].

Let $N \equiv N(R)$ be a random number of galaxies occurring inside of a sphere $V(R)$ of the radius R centered at the origin. Then factorial moments

$$\langle N^{[k]}(R)\rangle = \int\limits_{V(R)} dx_1 \; ... \int\limits_{V(R)} dx_k f_k(x_1, ..., x_n).$$

Using this representation, we obtain expressions for ordinary moments of the random number $N(R)$ called all count [18]:

$$\langle N(R)\rangle = nV, \quad V = (4\pi/3)R^3, \qquad (3.1)$$

$$\langle N^2(R)\rangle = nV + (nV)^2 + \Xi_2(R), \qquad (3.2)$$

$$\langle N^3(R)\rangle = nV + 3(nV)^2 + (nV)^3 + 3(nV+1)\Xi_2(R) + \Xi_3(R), \quad (3.3)$$

and so on. The functions $\Xi_k(R)$ are expressed through integrals of correspondent correlation functions $\xi_k(\mathbf{x}_1, \dots, \mathbf{x}_k)$:

$$\Xi_k(R) = n^k \int\limits_{V(R)} d\mathbf{x}_1 \dots \int\limits_{V(R)} d\mathbf{x}_k \xi_k(\mathbf{x}_1, \dots, \mathbf{x}_k). \qquad (3.4)$$

As one can see from above the variance of the cell count

$$DN(R) \equiv \langle N^2(R)\rangle - \langle N(R)\rangle^2 = nV + \Xi_2(R) \quad (3.5)$$

is essentially positive value.

The second kind of statistics – conditional statistics – is based on the weak version of the Cosmological Principle implying that all the *mass points (galaxies)* should be statistically equivalent with respect to their environment. The corresponding conditional factorial densities are defined under condition that the origin of coordinates is choosing at a galaxy. We will denote them $\overset{\circ}{f}_k(\mathbf{x}_1, \dots, \mathbf{x}_k)$.

Using the function, one can determine factorial moments of number of galaxies $\overset{\circ}{N}(R)$ in a spherical neighbourhood $V(R)$ of a randomly chosen galaxy

$$\langle \overset{\circ}{N}(R)\rangle = nV(R) + \Xi_2(R).$$

The correlation integral Ξ_2 generates the difference between conditional and unconditional mean numbers of galaxies inside a sphere, so called *cell counts* $\langle \overset{\circ}{N}(R)\rangle$ and $\langle N(R)\rangle = nV(R)$ respectively. Note that the conditional number density $\overset{\circ}{n}(r)$ is determined via relation

$$\overset{\circ}{f}_1(x) \equiv \overset{\circ}{n}(r) = \frac{d\langle \overset{\circ}{N}(r)\rangle}{4\pi r^2 dr} = ns(r), \qquad (3.6)$$

where

$$s(r) = 1 + \xi(r) \qquad (3.7)$$

is called sometimes the *structure function* [19]. If $s > 1$ (ξ is positive), we have an excess probability over the Poissonian case and, therefore, clustering.

If $s < 1$ (ξ is negative), we meet anticlustering when the probability to find a galaxy in a neighborhood of another one is less than in the Poissonian ensemble. The distance r_0 at which $\xi = 1$ is called the *correlation length* and this implies that there should be no appreciable overdensities (clusters) and underdensities (voids) extending over distances essentially larger than r_0.

4. Fractal concept

Another point of view is based on the assumption that the Universe reveals self-similarity with respect to scale, so-called *scaling*. Naturally, an infinite homogeneous medium possesses this property: it is homogeneous at all scales. However, there exist non-homogeneous self-similar systems called *fractals*, in which more and more structures appear at larger and larger scales and all the structures are similar to the one at small scales. In Fig. 1 we can see an elementary fractal distribution of points in space whose construction is evident. Starting from a point occupied by an object and counting how many objects are present within a volume characterized by a certain length scale, we get $\overset{\circ}{N}_1$ point objects within a radius R_1, $\overset{\circ}{N}_2 = q \overset{\circ}{N}_1$ objects within a radius $R_2 = kR_1$, $\overset{\circ}{N}_3 = q \overset{\circ}{N}_2 = q^2 \overset{\circ}{N}_1$ objects within $R_3 = kR_2 = k^2 R_1$ and so on. In general we have

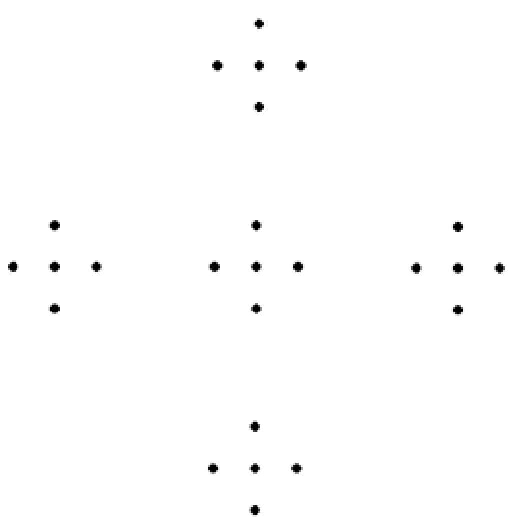

Figure 1. A simple deterministic fractal with fractal dimension $D = 1.2$

$$\overset{\circ}{N}_n \,/\, \overset{\circ}{N}_1 = q^{n-1}$$

and

$$R_n/R_1 = k^{n-1},$$

where q and k are constants. By taking the logarithm of the equations and dividing one by the other we get

$$\overset{\circ}{N}_n = BR_n^D \tag{4.1}$$

with

$$B = \overset{\circ}{N}_1 \, R_1^{-D},$$

$$D = \frac{\ln q}{\ln k},$$

where B is a prefactor of proportionality related to the lower cutoffs $\overset{\circ}{N}_1$ and R_1 of the fractal system, that is, the inner limit where the fractal system ends, and D is the fractal dimension($D < 3$). If we smooth out the point structure we get the continuum limit of equation (4.1)

$$\overset{\circ}{N}(R) = BR^D.$$

Further generalization of the fractal idea has led to the concept of *random* or *stochastic fractals*. In the frame of this approach, *(i)* $\overset{\circ}{N}(R)$ is a random variable with the mean value

$$\langle \overset{\circ}{N}(R) \rangle = BR^D,$$

(ii) the normalized random variable $Z = \overset{\circ}{N}(R)/\langle \overset{\circ}{N}(R)\rangle$ has the same distribution at all radiuses, i.e. a fractal structure is always connected with large fluctuations and clustering at all scales, and *(iii)* all its points are statistically equivalent with respect to their environments, there is no privileged points among them.

The density averaged over a sphere $V(R)$

$$\langle \overset{\circ}{n} \rangle_V \equiv \langle \overset{\circ}{N}(R)\rangle / V(R) = (3/4\pi) B R^{-\gamma}, \ \gamma = 3 - D$$

tends to zero as $R \to \infty$ (global number density) and diverges as $R \to 0$ (local number density). Therefore, one can use neither the densities nor correlation functions in the fractal case. Instead of them, the conditional density is used

$$\overset{\circ}{n}(r) = [d\langle \overset{\circ}{N}(r)\rangle/dr]/S(R) = (BD/4\pi) r^{-\gamma} \tag{4.2}$$

where $S(r)$ is the area of a spherical shell of radius r. Usually, the exponent $\gamma = 3 - D$ that defines the decay of the conditional density is called the *codimension*.

5. Observed characteristics

Direct calculations from catalogues show that in wide region of distances ξ can be described by means of inverse power law [3, 20] :

$$\xi(r) = (r/r_0)^{-\gamma}, \quad 0,06 < r \leq 60 \quad \text{Mpc}, \quad (5.1)$$

where

$$\gamma = 1.77 \pm 0.04$$

and r_0 is the *correlation length* varying in a wide region from sample to sample.

The third order correlation function obtained from observations has been approximated by the sum of products

$$\xi_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = Q\left[\xi(r_{12})\xi(r_{23}) + \xi(r_{31})\xi(r_{12}) + \xi(r_{23})\xi(r_{31})\right] \stackrel{s}{=} 3Q\xi(r_{12})\xi(r_{23}),$$

where $r_{ij} = |\mathbf{x}_i - \mathbf{x}_j|$ and Q is a constant of order 1 (from 0.80 ± 0.07 to 1.29 ± 0.21 for different samples [3]). The function is used for evaluation of the mean-square relative velocities of galaxies.

The measurement of higher order correlation functions $\xi_k, k > 3$ indicates that they can also be decomposed in such a way at least up to $n = 8$ ([18, 21–24]). This hierarchical model manifests the scaling property

$$\xi_k(\lambda\mathbf{x}_1, \ldots, \lambda\mathbf{x}_k) = \lambda^{-(k-1)\gamma}\xi_k(\mathbf{x}_1, \ldots, \mathbf{x}_k),$$

which is also confirmed by the outputs of cosmological N-body simulations. From the theoretical point of view, the hierarchical behaviour of correlation functions arises from a first-order perturbative approach to the evolution density inhomogeneities [25], and is consistent with models of non-linear clustering, such as the solution of the BBGKY equations in the strongly non-linear regime [26, 27] and the termodynamical approach proposed by Saslaw et al [28, 29].

It is often useful to analyze the statistics of the galaxy distribution in Fourier space, instead of in configuration space , as done by correlation functions. The Fourier transform $P(k)$ of two-point correlation function $\xi(r)$ is called the *power spectrum* [3]:

$$P(k) = \int \xi(r) \mathrm{e}^{i\mathbf{k}\mathbf{x}} d\mathbf{x}, \quad k = |\mathbf{k}|.$$

The power spectrum obtained on data from Lick Observatory catalog [30] cited in [31] is shown in Fig.2 (k in units of “waves per box”, physical wavelengths are $\lambda = 260h^{-1}\text{Mpc}/k$).

There exists several approximations of power spectrum. The simplest of them is

$$P(k) = Ak^{-\alpha}, \quad A = \text{const}. \quad (5.1)$$

It corresponds to the correlation function with $\gamma = 3 - \alpha$ and

$$r_0^\gamma = (A/2\pi^2)\Gamma(2-\alpha)\sin(\alpha\pi/2).$$

6. From power spectrum to random walks

As one can see from Fig. 2, the formula (5.1) is in agreement with observed results in the region $3 \leq k \leq 30$, but it may be improved in the region of small values of k which affects the $\xi(r)$ at large distances. Obviously, there are many appropriate representations of it, but one of them leads us to a very useful and significant analogy:

$$P(k) = A/[\mathrm{e}^{(bk)^\alpha} - c], \quad (6.8)$$

where $b > 0$ and $0 \leq c \leq 1$ can be chosen in an appropriate way. The point is that the inverse Fourier transformation of (6.1) gives the integral equation

$$\xi(r) = Ap(r/b^3) + (c/b^3) \int \xi(|\mathbf{x} - \mathbf{x}'|) p(r\mathbf{x}'/b) d\mathbf{x}', \quad (6.9)$$

where

$$p(\mathbf{x}) = q(\mathbf{x}; \alpha) \equiv (2\pi)^{-3} \int \mathrm{e}^{-i\mathbf{k}\mathbf{x} - k^\alpha} d\mathbf{k}. \quad (6.10)$$

Eq. (6.2) is the Ornstein-Zernike equation [32]. It is derived for such random media as physical liquids and gases consisting from atoms and molecules and exhibits the relationship between statistical mechanics and the concepts under consideration. On the other hand, this equation leads directly to the random walk model as a tool for construction of random point distribution with given correlation function ξ.

As follows from (5), the function

$$g(\mathbf{x}) = A^{-1}\xi(b|\mathbf{x}|)$$

obeys the equation

$$g(\mathbf{x}) = p(\mathbf{x}) + c \int g(\mathbf{x} - \mathbf{x}') p(\mathbf{x}') d\mathbf{x}', \quad (6.11)$$

where

$$p(\mathbf{x}) \geq 0, \quad \int p(\mathbf{x}) d\mathbf{x} = 1, \quad 0 \leq c \leq 1.$$

Its solution can be interpreted as a density of collisions of some particle starting its moving from the

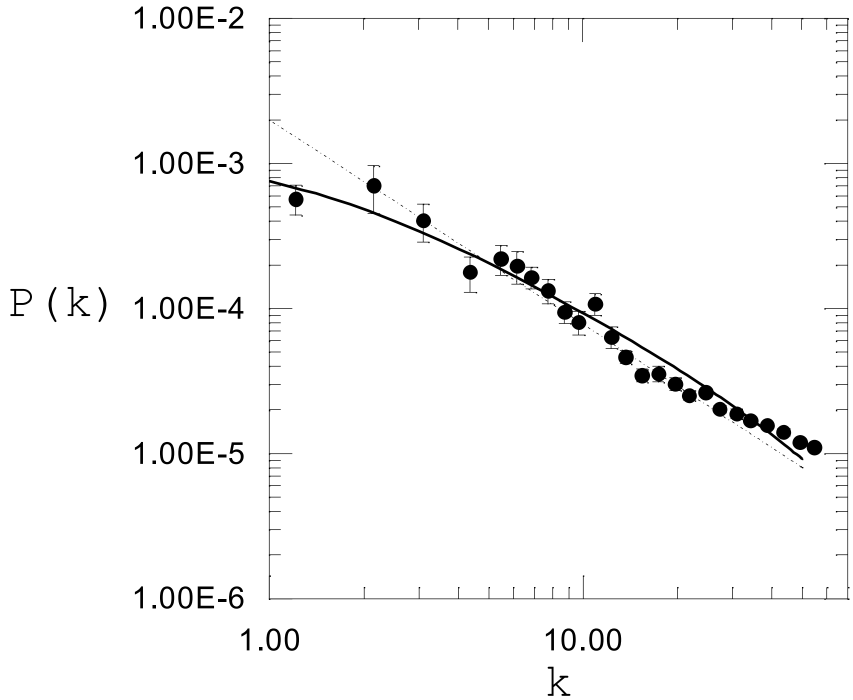

Figure 2. Power spectrum $P(k)$ (dots with error bars). The solid curve is approximate formula for $\alpha = 1.5$, $b = 0.018$, $A = 10^{-5}$ and $c = 0.99$. The dashed curve corresponds to the law $P(k) \propto k^{-1.41}$.

origin $\mathbf{X}_0 = 0$ and performing the first collision at the point $\mathbf{X}_1 \in d\mathbf{x}'$ with probability $p(\mathbf{x}')d\mathbf{x}'$. Here it stops with the probability $1-c$ or performs the next jump into $\mathbf{X}_2 \in d\mathbf{x}''$ with the probability $cp(\mathbf{x}'' - \mathbf{x}')d\mathbf{x}''$ and so forth.

The integral

$$\int g(\mathbf{x})d\mathbf{x} = 1/(1-c), \quad c < 1, \tag{6.12}$$

gives the mean number of all collisions of the particle including the final one, and the function $g(\mathbf{x})$ itself can be expressed by the Neumann series expansion

$$g(\mathbf{x}) = \sum_{j=1}^{\infty} c^{j-1} p^{\star j}(\mathbf{x}),$$

where

$$p^{\star 1}(\mathbf{x}) \equiv p(\mathbf{x})$$

and

$$p^{\star (j+1)}(\mathbf{x}) = \int p^{\star j}(\mathbf{x} - \mathbf{x}')p(\mathbf{x}')d\mathbf{x}'$$

is the j-fold convolution of the distribution density $p(\mathbf{x})$.

The GF of the random set of nodes of a single trajectory obeys the integral equation

$$G(\mathbf{x} \to u(\cdot)) = \int d\mathbf{x}' p(\mathbf{x} \to \mathbf{x}')[(1-c)u(\mathbf{x}') + cu(\mathbf{x}')G(\mathbf{x}' \to u(\cdot))]. \tag{6.13}$$

For the sake of most clearness we write here and below $p(\mathbf{x} \to \mathbf{x}')$ and $g(\mathbf{x} \to \mathbf{x}')$ instead of $p(\mathbf{x}' - \mathbf{x})$ and $g(\mathbf{x}' - \mathbf{x})$ respectively.

Multi-fold function differentiating with respect to $u(\mathbf{x})$ yields the integral equation

$$g_k(\mathbf{x} \to \mathbf{x}_1, \ldots, \mathbf{x}_k) \stackrel{s}{=} k! cp(\mathbf{x} \to \mathbf{x}_1) g_{k-1}(\mathbf{x}_1 \to \mathbf{x}_2, \ldots, \mathbf{x}_k) + c \int d\mathbf{x}' p(\mathbf{x} \to \mathbf{x}') g_k(\mathbf{x}' \to \mathbf{x}_1, \ldots, \mathbf{x}_k).$$

Multiplying both sides of Eq.(6.4) by $kcg_{k-1}(\mathbf{x}_1 \to \mathbf{x}_2, \ldots, \mathbf{x}_k)$ and comparing the result with the above equation, we see, that

$$g_k(\mathbf{x} \to \mathbf{x}_1, \ldots, \mathbf{x}_k) \stackrel{s}{=} kcg_1(\mathbf{x} \to \mathbf{x}_1) g_{k-1}(\mathbf{x}_1 \to \mathbf{x}_2, \ldots, \mathbf{x}_k) \stackrel{s}{=} k! c^{k-1} g_1(\mathbf{x} \to \mathbf{x}_1) g_1(\mathbf{x}_1 \to \mathbf{x}_2) \ldots g(\mathbf{x}_{k-1} \to \mathbf{x}_k). \tag{6.14}$$

7. Poisson ensemble of independent trajectories

Now we shall consider an infinite set of independent random trajectories starting from different random points of birth distributed according to Poisson uniform law:

(i) the random numbers of the points $N_0(V_1)$ and $N_0(V_2)$ belonging to arbitrary disjoint domains V_1 and V_2 respectively are mutually independent;

(ii) the random variable $N_0(V)$ obeys the Poisson distribution with the mean value $\langle N_0(V)\rangle$ for an arbitrary V;

(iii) the mean value of the variable is directly proportional to the volume independently of its shape, $\langle N_0(V)\rangle = n_0 V$.

Under such conditions, the GF of unconditional random measure $N(V)$ takes the form

$$F(u(\cdot)) = \exp\left\{n_0 \int [G(\mathbf{x} \to u(\cdot)) - 1] d\mathbf{x}\right\}, \tag{7.15}$$

where an auxiliary function $u(\mathbf{x})$ must be chosen in the way the integral converges. As a result (see (6.5)) we have

$$f_1 = \frac{n_0}{1-c} = \xi_k(\mathbf{x}_1, ..., \mathbf{x}_k) = n^{-k} n_0 \int g_k(\mathbf{x} \to \mathbf{x}_1, ..., \mathbf{x}_k) d\mathbf{x} \stackrel{s}{=} 2^{1-k} k! \xi(r_{12})\xi(r_{23})\dots\xi(r_{k-1,k}), \tag{7.16}$$

where

$$\xi(r_{ij}) = 2(c/n) g(\mathbf{x}_i \to \mathbf{x}_j) = 2[c(1-c)/n_0] g(\mathbf{x}_i \to \mathbf{x}_j) \tag{7.17}$$

and we come to the Ornstein-Zernike equation for $\xi(r)$ again.

In case of conditional statistics the GF $\overset{\circ}{F}(u(\cdot))$ of the conditional random measure $\overset{\circ}{N}(V)$ is of the form

$$\overset{\circ}{F}(u(\cdot)) = W(u(\cdot)) \exp\left\{n_0 \int [G(\mathbf{x} \to u(\cdot)) - 1] d\mathbf{x}\right\}, \tag{7.18}$$

where $W(u(\cdot))$ is the GF of the trajectory a node of which is chosen as the origin of coordinates. If the trajectory is very long (strictly speaking, infinite) it can be decomposed into two independent trajectories starting from the origin of coordinates. In this case

$$W(u(\cdot)) = G^2(u(\cdot)) \tag{7.19}$$

Functional differentiation yields

$$\overset{\circ}{f}_1(\mathbf{x}_1) = w_1(\mathbf{x}_1) + n_0 \int g(\mathbf{x} \to \mathbf{x}_1) d\mathbf{x}, \tag{7.20}$$

$$\overset{\circ}{f}_2(\mathbf{x}_1, \mathbf{x}_2) \stackrel{s}{=} w_2(\mathbf{x}_1, \mathbf{x}_2) + 2n_0 w_1(\mathbf{x}_1) \int g(\mathbf{x} \to \mathbf{x}_2) d\mathbf{x}' + n_0 \int g(\mathbf{x} \to \mathbf{x}_1) d\mathbf{x} \cdot n_0 \int g(\mathbf{x} \to \mathbf{x}_2) d\mathbf{x} + n_0 \int g_2(\mathbf{x} \to \mathbf{x}_1, \mathbf{x}_2) d\mathbf{x} \tag{7.21}$$

and so one. Here

$$w_1(\mathbf{x}_1) = \left.\frac{\delta W(u(\cdot))}{\delta u(\mathbf{x}_1)}\right|_{u=1},$$

$$w_2(\mathbf{x}_1, \mathbf{x}_2) = \left.\frac{\delta^2 W(u(\cdot))}{\delta u(\mathbf{x}_2)\delta u(\mathbf{x}_1)}\right|_{u=1},$$

and so one are factorial moment densities of the set of nodes of the proper trajectory passing through the, origin of coordinates. Taking into account (6.5) and (7.2), we obtain from Eq. (7.6) that

$$w_1(\mathbf{x}) = \xi(r)/n, \qquad r = |\mathbf{x}|.$$

Further,

$$\overset{\circ}{\varphi}_2(\mathbf{x}_1, \mathbf{x}_2) = \overset{\circ}{f}_2(\mathbf{x}_1, \mathbf{x}_2) - \overset{\circ}{f}_1(\mathbf{x}_1) \overset{\circ}{f}_1(\mathbf{x}_2) = [w_2(\mathbf{x}_2, \mathbf{x}_2) - w_1(\mathbf{x}_1) w_1(\mathbf{x}_2)] + n_0 \int g_2(\mathbf{x} \to \mathbf{x}_1, \mathbf{x}_2) d\mathbf{x} = \theta_2(\mathbf{x}_1, \mathbf{x}_2) + n\xi(r_{12}),$$

where

$$\theta_2(\mathbf{x}_1, \mathbf{x}_2) = w_2(\mathbf{x}_1, \mathbf{x}_2) - w_1(\mathbf{x}_1) w_1(\mathbf{x}_2).$$

Obviously,

$$\overset{\circ}{f}_1(\mathbf{x}) \sim \begin{cases} w_1(\mathbf{x}) & \text{at small distances,} \\ n = n_0/(1-c) & \text{at large distances.} \end{cases}$$

Such conclusion can be made with respect to any characteristics of the distribution, and one can consider that

$$\overset{\circ}{F}(u(\cdot)) \sim \begin{cases} W(u(\cdot)), & \text{if } u(\mathbf{x}) \text{ differs from 1} \\ & \text{at small } |\mathbf{x}|, \\ F(u(\cdot)), & \text{if } u(\mathbf{x}) \text{ differs from 1} \\ & \text{at large } |\mathbf{x}|. \end{cases}$$

The two extreme cases will be considered below. But beforehand, we have to specify the walking process.

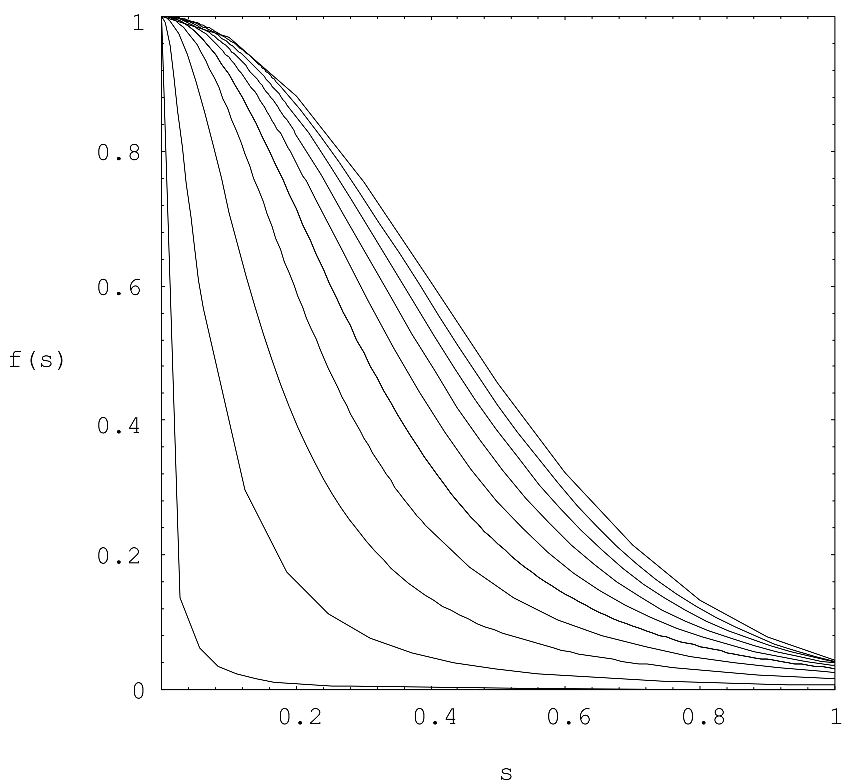

Figure 3. The renormalized three-dimensional spherically symmetric density $f(s) \equiv c^{-3}q(c^{-1}s;\alpha)$, $c^3 = q(0;\alpha)$ for different characteristic exponents $\alpha = 0.2(0.2)2.0$.

8. The Lèvy-Feldheim distributions

The walk with transition probability of asymptotically inverse power type $p(\mathbf{x}) \propto r^{-\gamma}$is called the Rayleigh - Lèvy walk. The three-dimensional Lèvy- Feldheim distributions [33–35]

$$p(\mathbf{x}) \equiv q(\mathbf{x}; \alpha), \alpha \in (0, 2].$$

appeared in Section 6 belong to this class. One of them, with $\alpha = 2$, is the famous Gaussian law, and another one, $q(x; 1) = [\pi(1 + r^2)]^{-2}$ is the Cauchy law. All other stable densities can not be expressed in terms of elementary functions but can be calculate by means of series

$$q(\mathbf{x}, \alpha) = \frac{1}{2\pi^2\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \Gamma((2n+3)/\alpha) r^{2n},$$

or

$$q(\mathbf{x}, \alpha) = \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \Gamma(n\alpha + 2) \sin(n\alpha\pi/2) r^{-n\alpha-3}.$$

In some cases, the integral representation of the densities is more useful. As an example, we write the density with $\alpha = 3/2$:

$$q(\mathbf{x}; 3/2) = (4\pi r^2)^{-1} H(r),$$

where

$$H(r) = \frac{2}{\pi r} \int_0^{\infty} e^{-(x/r)^{3/2}} x \sin x dx$$

is Holtsmark's distribution, describing gravitational force fluctuations in the homogeneous Poisson ensemble of point sources [36, 37].

Some graphs of the densities are plotted in Fig 3–4.

Two properties of the stable laws are very important for the problem under consideration. Firstly, multiple convolutions of this density are expressed in terms of the original density with rescaled argument:

$$q^{\star j}(\mathbf{x}; \alpha) = j^{-3/\alpha} q(\mathbf{x} j^{-1/\alpha}; \alpha),$$

and, secondly, the stable density with $\alpha < 2$ has the inverse power tail at large distances,

$$q(\mathbf{x}; \alpha) \sim C(\alpha) r^{-\alpha-3}, \quad r \to \infty$$

where

$$C(\alpha) = (2\pi^2)^{-1} \Gamma(2 + \alpha) \sin(\pi\alpha/2).$$

At large distances

$$q^{\star j}(\mathbf{x}; \alpha) \sim jC(\alpha) r^{-\alpha-3}, \quad r \to \infty.$$

9. The Rayleigh-Lèvy walk

Using the above results we arrive at the expression

$$g(\mathbf{x}) = (2\pi^2)^{-1} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \Gamma(n\alpha+2) \sin(n\alpha\pi/2) \Phi(c, -n, 1) r^{-3-n\alpha}, \quad c < 1, \quad (9.22)$$

where

$$\Phi(c, -n, 1) = \sum_{k=1}^{\infty} c^{k-1} k^n.$$

The asymptotic behaviour of (9.1) far away from origin is governed by the first term of the series:

$$g^{as}(\mathbf{x}) \sim C(\alpha)(1-c)^{-2} r^{-3-\alpha}, \qquad C < 1. \quad (9.23)$$

As $c \to 1$

$$\Phi(c, -n, 1) \sim n!(1 - c)^{-1-n}$$

and

$$g(\mathbf{x}) \sim (2\pi^2)^{-1} \sum_{n=1}^{\infty} (-1)^{n-1} \Gamma(n\alpha+2) \sin(n\alpha\pi/2) (1 - c)^{-1-n} r^{-3-n\alpha}, \quad c \to 1.$$

However this result is not applicable to the case $c = 1$, which should be considered separately. Putting $c = 1$ in Eq. (6.4) and applying the Euler-Maclaurin summation formula, we obtain the asymptotic expansion

$$\left. \begin{aligned} g^{as}(\mathbf{x}) &= |C(-\alpha)| r^{-3+\alpha} + \\ (2\pi^2)^{-1} &\sum_{n=1(\text{odd})}^{\infty} \Gamma(n\alpha+2) \sin(n\alpha\pi/2) A_n r^{-n\alpha-3} \end{aligned} \right\}$$

where

$$A_n = \frac{1}{n!2} - \frac{1}{(n+1)!} - \sum_{m=1}^{(n+1)/2} \frac{B_{2m}}{(2m)!(n-2m+1)!}.$$

In particular $A_1 = -1/12$, $A_3 = 1/720$, $A_5 = -1/30240$, $A_7 = 1/1209600$, $A_9 = -1/47900160$ and so on.

Thus, in case of infinite trajectories we have the opposite sign before α

$$g^{(as)}(\mathbf{x}) = |C(-\alpha)| r^{\alpha-3}, \quad c = 1, \qquad (9.3)$$

and we deal with a stochastic fractal of dimensional $D = \alpha \in (0, 2]$. The difference between asymptotics (9.2) and (9.3) produces the question: can one simulate stochastic fractals with finite trajectories? An

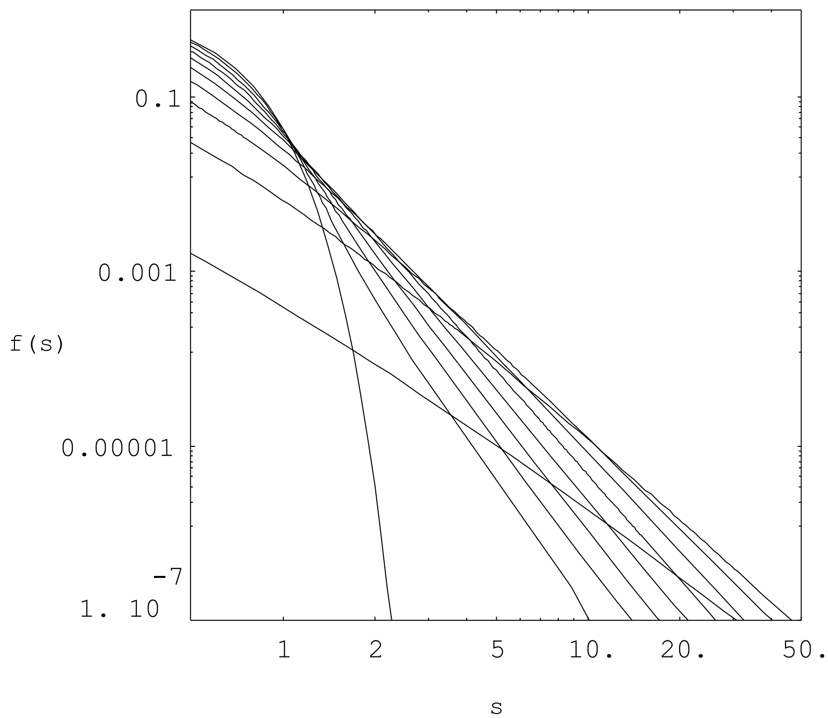

Figure 4. Same as in Fig.3 for $s \in [0.5, 50]$

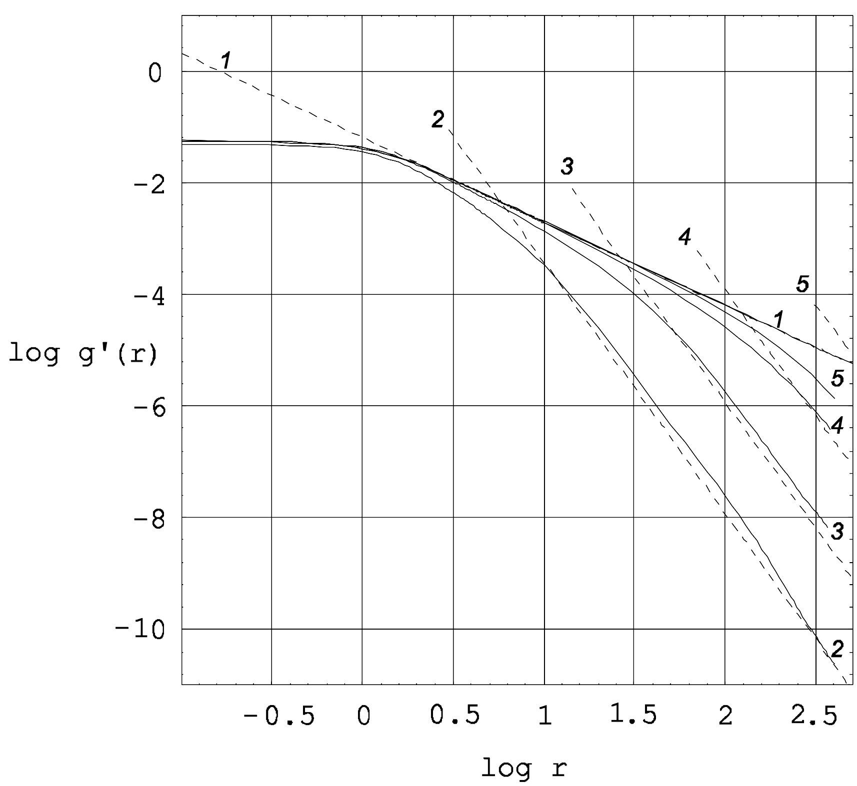

Figure 5. The function $g(r)$ for $\alpha = 1.5$ and various values of c: 1 (line 1), 0.9 (line 2), 0.99 (line 3), 0.999 (line 4) and 0.9999 (line 5). Dotted lines show main asymptotic terms.

answer to question is given by numerical calculations. Their results show that in case when c is close to 1, there exists some region of distances where $g_c(\mathbf{x})$ practically coincides with $g_1(\mathbf{x})$, and the closer c to 1 the wider this region (see Fig.5). This phenomenon was named *intermediate asymptotics*.

Passing to the dimensional space variable $r^* = br$, $n_0^* = b^3 n_0$ one can represent the dimensionaless correlation function $\xi(r)$ in the form (see (7.3))

$$\xi(r^*) = [2c(1-c)/n_0^*]g(r^*/b) \sim (r^*/r_0^*)^{-\gamma}, \quad (9.4)$$

where $\gamma = 3 - \alpha$ and

$$r_0^{*\gamma} = [2c(1-c)/n_0^*]|C(-\alpha)|b^\gamma. \quad (9.5)$$

Thus, the model under considerations contains four numerical parameters (α, c, n_0^* and b) which completely determinate correlation functions of all orders.

10. Simulation of RL fractal

The algorithm of the RL-walk simulation looks as follows: the first galaxy is placed at some given point, say at the origin of coordinates, the radius-vector of the second galaxy is chosen from the isotropic inverse power distribution, the position of every next galaxy relative to the previous one is sampling from the same distribution. The obtained infinite sequence $\{0, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, ...\}$ can be considered as a set of collision points of a walker moving through some hypothetical medium in straight lines between collisions and changing its direction in collision events isotropically.

For simulation of 3-dimensional random vectors $\mathbf{S}(\alpha)$ with isotropic Lèvy-Feldheim distribution $q(\mathbf{x};\alpha)$, the following algorithm can be used:

$$\mathbf{S}(\alpha) = \sqrt{S_+(\alpha/2)}\mathbf{S}(2)$$

where $S_+(\alpha/2)$ is the *subordinator* [35], i.e. an independent of $\mathbf{S}(\alpha)$ random variable with the one-sided stable density $q_+(x;\alpha/2)$ whose Laplace transform

$$\widetilde{q}_+(\lambda;\alpha/2) \equiv \int_0^\infty q_+(x;\alpha/2)\exp(-\lambda x)dx = \exp(-\lambda^{\alpha/2}).$$

Since

$$\langle \exp i\mathbf{k}\mathbf{S}(2)\rangle = \exp(-k^2)$$

vector $\mathbf{S}(2)$ can be expressed through the 3-dimensional isotropic Gaussian vector $\mathbf{N}$ via the relation $\mathbf{S}(2) = \sqrt{2}\mathbf{N}$. Different components $N_i (i = 1, 2, 3)$ of the vector $\mathbf{N}$ are mutually independent and can be simulated in the ordinary way:

$$N_i = \sqrt{2E}\cos(2\Theta)$$

where $E = -\ln\gamma_1, \Theta = \pi\gamma_2$, γ_j are independent uniformly distributed on (0,1) random variables.

Simulation of subordinators can be performed by means of Kanter's algorithm [38]:

$$S_+(\alpha) = [U_\alpha(\Theta)/E]^{(1-\alpha)/\alpha}.$$

with

$$U_\alpha(\Theta) = \frac{[\sin(\alpha\Theta)]^{\alpha/(1-\alpha)}\sin((1-\alpha)\Theta)}{(\sin\Theta)^{1/(1-\alpha)}}.$$

Let return to the RL trajectory. As mentioned in Sec. 4, all points of a stochastic fractal have to be statistically equivalent with respect to their environment. The above structure does not meet the requirement. To see this, let us denote the density of collision points for a trajectory with a finite number of steps m by $g^{(m)}(\mathbf{x})$. Using the notation we can write down the density of all points relative to the m-th point of the infinite trajectory in the form:

$$g_1(\mathbf{x}|\mathbf{x}_m = 0) = g_1(\mathbf{x}) + g^{(m)}(\mathbf{x}), \quad m \ge 0,$$

where $g^{(m)}(\mathbf{x}) \to g_1(\mathbf{x})$ as $m \to \infty$.

Hence, the structure function changes from $g_1(\mathbf{x})$ to $2g_1(\mathbf{x})$ while m runs from 0 to ∞.

To avoid the privileged positions we have offered to supplement RL trajectory in the symmetrical (in the statistical sense) way. As a result, we have got an infinite "to both sides" trajectory: $\{\cdots, \mathbf{X}_{-3}, \mathbf{X}_{-2}, \mathbf{X}_{-1}, 0, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3 \cdots\}$. All the points are statistically equal now, each point of the trajectory has the same structure function $f_1(\mathbf{x}) = 2g_1(\mathbf{x})$ on any scale. Because of spherical symmetry of the walk, the "left-hand" part of the trajectory $\{\cdots, \mathbf{X}_{-3}, \mathbf{X}_{-2}, \mathbf{X}_{-1}\}$ can be considered as an independent (in the statistical sense) copy of the "right hand" part $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \cdots\}$ and constructed in the same way. We call this construction *a paired RL trajectory*.

Set of nodes of the trajectories we will call the *RL fractal* and the *paired RL fractal* respectively.

Note, that RL and paired RL trajectories yield stochastic fractals only with fractal dimension $D < 2$. To obtain fractals with $D > 2$, one have to use branching RL trajectories [44]. In this case, $D = 2\alpha, \alpha \in (1, 3/2)$.

11. Cell count on small scalles

As shown in Sec. 7, GF at small scales is of the form

$$\overset{\circ}{F}(u(\cdot)) = G^2(0 \to u(\cdot))$$

Under condition $c = 1$ the GF $G(\mathbf{x} \to u(\cdot))$ obeys the equation

$$G(\mathbf{x} \to u(\cdot)) = \int d\mathbf{x}' p(\mathbf{x} \to \mathbf{x}') u(\mathbf{x}') G(\mathbf{x}' \to u(\cdot)).$$

The k-th factorial moment of the random cell-count $\overset{\circ}{N}(R)$ is given by the formula

$$\langle \overset{\circ}{N}{}^{[n]}(R)\rangle = \sum_{k=0}^{n} \binom{n}{k} \langle \overset{\bullet}{N}{}^{[k]}(R)\rangle \langle \overset{\bullet}{N}{}^{[n-k]}(R)\rangle.$$

where

$$\langle \overset{\bullet}{N}{}^{[k]}(R)\rangle =$$
$$k! \int_{U_R} d\mathbf{x}_1 \ldots \int_{U_R} d\mathbf{x}_k g(\mathbf{x} \to \mathbf{x}_1) \ldots g(\mathbf{x}_{k+1} \to \mathbf{x}_k)$$

As $R \to \infty$ $\langle \overset{\bullet}{N}{}^{[k]}\rangle \sim \langle \overset{\bullet}{N}{}^{k}\rangle$. Using asymptotic expression (9.3) we have got

$$\langle \overset{\bullet}{N}{}^{k}(R)\rangle \sim k!(4\pi|C(-\alpha)|/3)^k R^{\alpha k} K_k(\alpha), \; R \to \infty,$$

Here

$$K_0(\alpha) \; = \; 1, \quad K_1(\alpha) \; = \; (3/4\pi) \int_{U_1} r^{-3+\alpha} d\mathbf{x},$$

$$K_k(\alpha) =$$
$$(3/4\pi)^k \int \!\!\ldots\!\! \int_{U_1^k} (r_{01} \ldots r_{k-1,k})^{-3+\alpha} d\mathbf{x}_1 d\mathbf{x}_2 \ldots d\mathbf{x}_k,$$
$$k > 1,$$

$$r_{ij} = |\mathbf{x}_i - \mathbf{x}_j|$$

and U_1 is the sphere of the radius $R = 1$ centered at the origin of coordinates.

Remind that the result. Numerical calculations of the multiple integrals has been performed in the work [39]. The results for are presented in Table 1.

As one can see from above the moments of the normalized cell-count $Y = \overset{\bullet}{N}(R)/\langle \overset{\bullet}{N}(R)\rangle$ (for a single trajectory) and $Z = \overset{\circ}{N}(R)/\langle \overset{\circ}{N}(R)\rangle$ (for a paired trajectory) do not depend on R as $R \to \infty$. As a result, the asymptotical distribution of $\overset{\circ}{N}(R)$ can be written in the scaling form:

$$\mathsf{P}\{Z = n/\langle \overset{\circ}{N}(R)\rangle\} \sim \frac{1}{\langle \overset{\circ}{N}(R)\rangle} \Psi_\alpha \left(\frac{n}{\langle \overset{\circ}{N}(R)\rangle} \right),$$
$$R \to \infty.$$

Here $\Psi_\alpha(z)$ is the density of the distribution of variable Z having the moments:

$$\langle Z^n \rangle = \int_0^\infty z^n \Psi_\alpha(z) dz, \quad \langle Z \rangle = 1,$$

numerical values of which are presented in Table 2.

As one can see from Table 2 the following relation holds true for the moments

$$\langle Z^n \rangle = (A(\alpha)n + B(\alpha)) \langle Z^{n-1} \rangle.$$

This relation is characteristic property for gamma distribution

$$\Psi_\alpha(z) = \frac{1}{\Gamma(\lambda)} \lambda^\lambda z^{\lambda-1} e^{-\lambda z}, \qquad (11.24)$$

whence we obtain

$$\langle Z^n \rangle = \left(\frac{n}{\lambda} + 1 - \frac{1}{\lambda} \right) \langle Z^{n-1} \rangle.$$

The parameter of gamma distribution λ can be expressed in terms of the variance of Z:

$$\lambda(\alpha) = \frac{1}{\langle Z^2 \rangle - 1} = \frac{1}{\sigma_Z^2}.$$

Note, that Eq. (11.1) is exactly the same formula derived in [40, 41] (see also [42]).

We have also performed Monte Carlo simulation of random trajectories with transition probability (5.5) for $\alpha = 0.5, 1.0, 1.5$. The point number count has been made inside the sphere of radius $R \gg a$. The trajectory is broken when coming out the boundary of the sphere of the radius $R_{max} = 1 \gg R$. As a result of simulation, the function $\Psi_\alpha(z)$ and its moments $\langle Z \rangle, \ldots, \langle Z^5 \rangle$ for α =0.5, 1.0, 1.5 have been obtained. In Fig. 6 the results of the simulation are shown for R =0.05, 0.1 and 0.2, α=1.0. All the histograms are satisfactory described by the gamma distribution.

Let emphasize that the fluctuations of point number count inside the sphere (centered around any galaxy) do not vanish with increasing the radius but stay an essential value up to some distance where the transition domain begins.

12. Cell count on large scales

As one can see from Eq. (7.8), on large scales both conditional and unconditional statistics coincide and we have

$$\overset{\circ}{\xi}(\mathbf{x}_1, ..., \mathbf{x}_k) = \xi_k(\mathbf{x}_1, ..., \mathbf{x}_k) \overset{s}{=}$$
$$2^{1-k} k! \xi(r_{12}) ... \xi(r_{k-1}, k).$$

Table 1

The coefficients $K_n(\alpha)$ for a single trajectory. The values in brackets are from the book [3]. The values with asterisks are the data for $\alpha = 1.23$ from the book [3].

K_n	α							
	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
K_1	12.0	6.00	4.00	3.00	2.40	2.00	1.71	1.50
K_2	140	34.0	14.6	7.80	4.88	3.31	2.42	1.88
		(34.0)		(7.82)	$(5.02)^*$	(3.31)		(1.88)
K_3	$163 \cdot 10$	185	50.4	19.1	9.38	5.22	3.30	2.29
					$(9.5)^*$			
K_4	$188 \cdot 10^2$	985	168	45.8	17.7	8.13	4.46	2.78
K_5	$215 \cdot 10^3$	$516 \cdot 10$	554	108	33.2	12.6	6.03	3.38

Table 2

The moments $\langle Z^n \rangle$ for a paired trajectory. The lower values are results of Monte Carlo simulation.

$\langle Z^n \rangle$	α						
	0.25	0.50	0.75	1.00	1.25	1.50	1.75
$\langle Z^2 \rangle$	1.49	1.44	1.41	1.37	1.35	1.33	1.33
		1.51 ± 0.07		1.40 ± 0.06		1.36 ± 0.06	
$\langle Z^3 \rangle$	2.93	2.70	2.55	2.36	2.29	2.22	2.22
		3.04 ± 0.25		2.52 ± 0.21		2.38 ± 0.21	
$\langle Z^4 \rangle$	7.18	6.18	5.59	4.95	4.71	4.51	4.54
		7.5 ± 1.0		5.7 ± 0.8		5.3 ± 0.9	
$\langle Z^5 \rangle$	21.0	16.7	14.4	12.2	11.4	10.8	11.0
		22 ± 4		16 ± 3		15 ± 5	

This means that no matter where we choose the origin of coordinate: at a galaxy or not. Therefore

$$\langle \overset{\circ}{N}{}^{[k]}(R) \rangle = \langle N^{[k]}(R) \rangle = \int\limits_{V(R)} d\mathbf{x}_1 ... \int\limits_{V(R)} d x_k f_k(\mathbf{x}_1, ..., \mathbf{x}_k),$$

and first ordinary moments are given via relations (3.1)–(3.4), where

$$\Xi_2(R) = n^2 \int\limits_{V(R)} d\mathbf{x}_1 \int\limits_{V(R)} d\mathbf{x}_2 \xi(r_{12}),$$

$$\Xi_3(R) = (3/2) n^3 \int\limits_{V(R)} d\mathbf{x}_1 \int\limits_{V(R)} d\mathbf{x}_2 \int\limits_{V(R)} d\mathbf{x}_3 \xi(r_{12}) \xi(r_{23})$$

and so forth. Using here formulas (7.3) and (9.3) written in the form (5.1) we obtain

$$\langle N^2(R) \rangle = nV + (nV)^2 + (nV)^2 A_2 (r_0/R)^\gamma \quad (12.25)$$

and

$$\langle N^3(R) \rangle = nV + 3(nV)^2 + (nV)^3 + 3(nV+1)(nV)^3 A_3 (r_0/R)^{2\gamma}, \quad (12.26)$$

where $A_k = 2^{1-k} k!$ and

$$J_k = \left(\frac{3}{4\pi} \right)^k \int\limits_{V(1)} dx_1 ... \int\limits_{V(1)} dx_k (r_{12} ... r_{k-1,k})^{-\gamma}.$$

The integrals J_2, J_3 and J_4 are numerically calculated in [3] (see Table 3).

Remind that the moments (12.1) and (12.2) were calculated by Peebles [3] and represented in the form (formulas (60.7) and (60.13) of the cited book)

$$\langle N^2(R) \rangle = (nV)^2 A'_2 (r_0/R)^\gamma \quad (12.3)$$

and

$$\langle N^3(R) \rangle = (nV)^3 A'_3 (r_0/R)^{2\gamma}. \quad (12.4)$$

Factors A'_k were obtained from observation data and from some simple model offered by Peebles (we

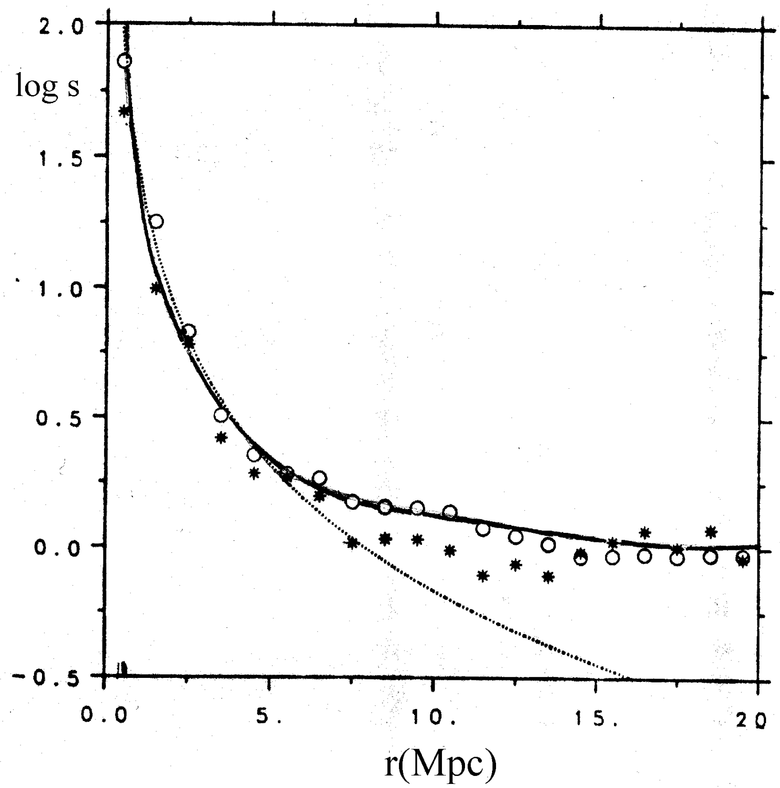

Figure 7. The structure function $s(r) = 1 + \xi(r)$, plotted as a function of linear scale r. The open circles and asterisks represent subsample S65 of the CfA redshift survey after different procedures for the edge correction (see [43]). The dotted line is the power law, the full line is a result of the mesofractal model.

will denote them by A''_k). The numerical results are presented in Table 3.

Peebles noticed that his results are bounded by the condition $R \le r_0$ because the variance

$$\mathsf{D}N(R) = \langle N^2(R)\rangle - \langle N(R)\rangle^2 = (nV)^2[A_2(r_0/R)^\gamma - 1]$$

must be positive. At the same time, his formule (60.3) coincides with our formula (12.1) and both

Table 3

Numerical values of integrals J_k and factors A_k.

k	2	3	4
J_k	1.82	3.38	6.3
A_k	1.82	5.07	18.9
A'_k	1.82	13.1	300
A''_k	2.10	8.40	60

they yield the variance

$$\mathsf{D}N(R) = (nV)^2[A_2(r_0/R)^\gamma + (nV)^{-1}]$$

being free from the defect. When $R \to \infty$

$$\mathsf{D}N(R) \sim nV = \langle N(R)\rangle$$

and

$$\langle N^3(R)\rangle \sim \langle N(R)\rangle + 3\langle N(R)\rangle^2 + \langle N(R)\rangle^3.$$

The two formulas are indication of the Poisson law. Consequently, there are reasons to suppose the galaxy distribution to be uniform at very large scales in accordance with the Cosmological Principle.

13. Transition from the fractal domain to the homogeneous region

According to CfA redshift survey, there is overwhelming evidence that the distribution of galaxies in the Universe obeys a power law on small values

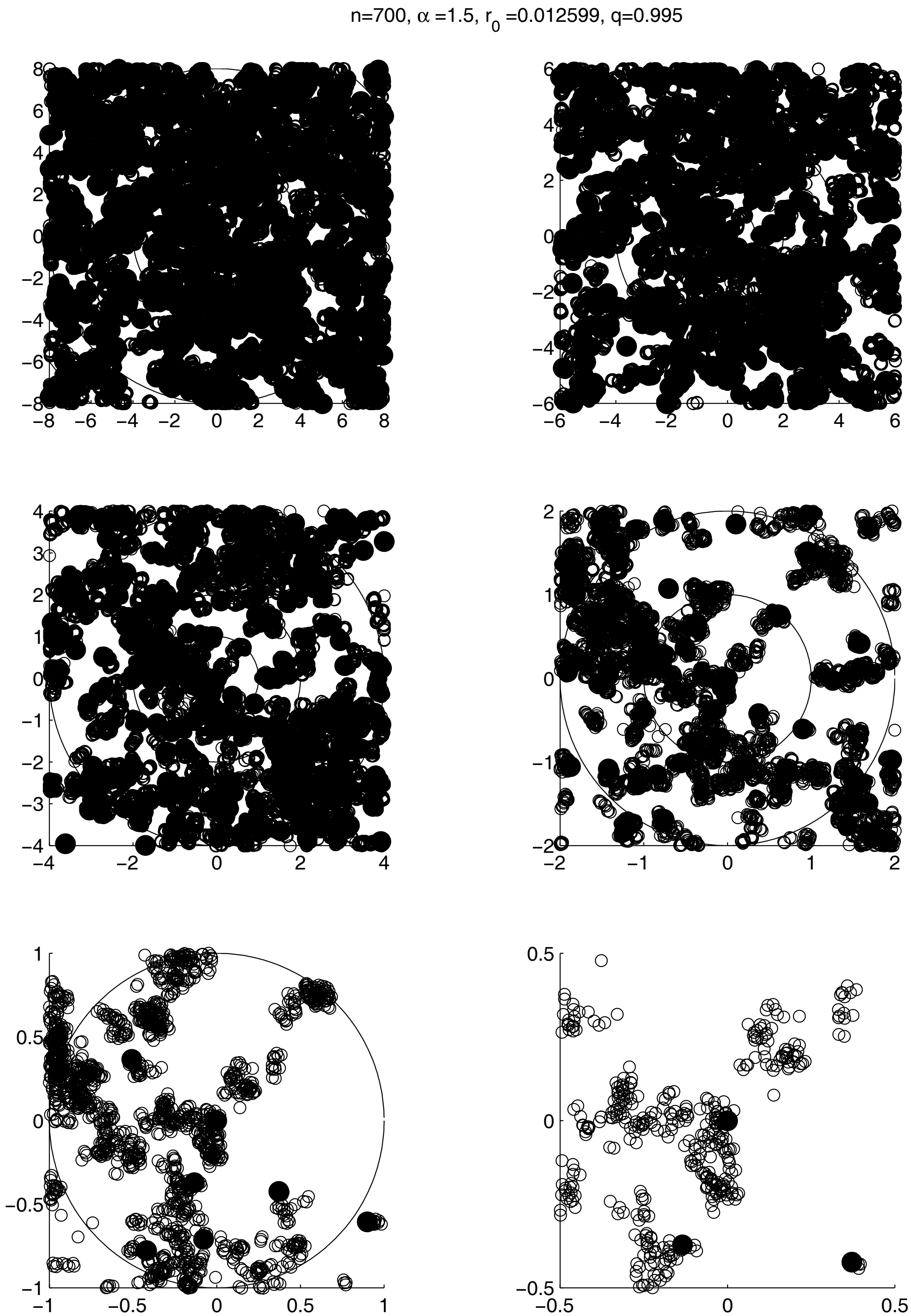

Figure 8. Results of simulation of the mesofractal embedded in a plane at different scales ($\alpha = 1.5$, number of trajectories $n = 700$, scale parameter $r_0 = 0.005$, survival probability $c = q = 0,995$).

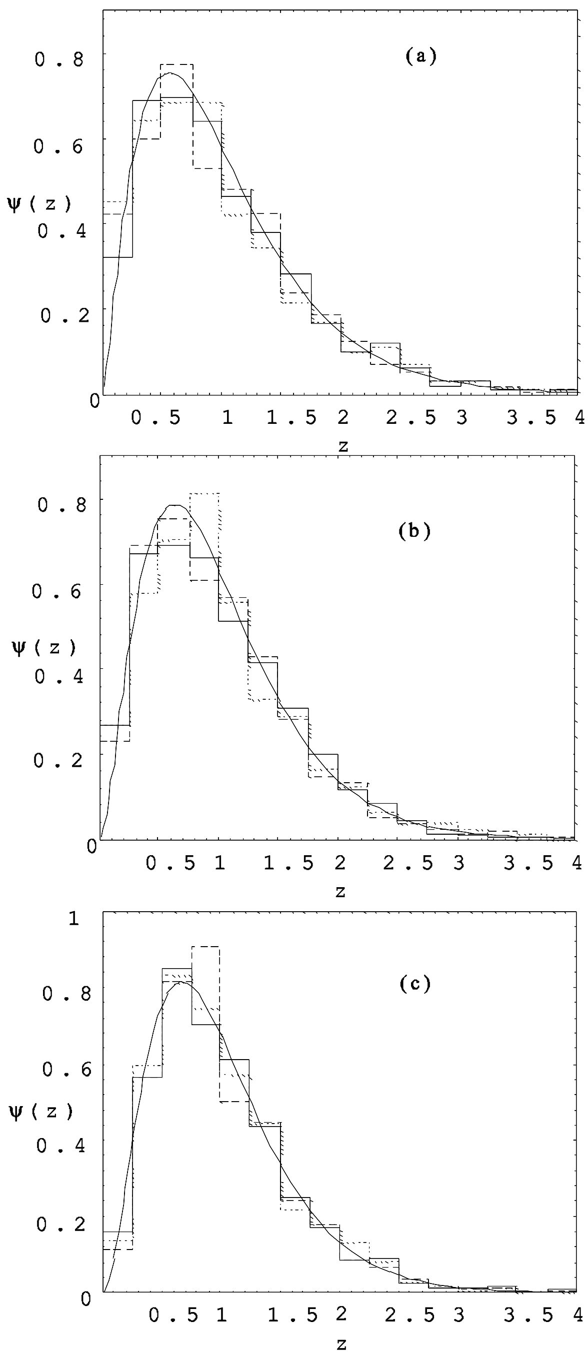

Figure 6. The probability density function $\Psi_\alpha(z)$ for $\alpha = 0.5$ (a), 1.0 (b) and 1.5 (c). Full lines show gamma approximation histograms mark results of Monte Carlo simulation for three different values of R (0.05, 0.1 and 0.2).

(about r_0) of distances and reveals a tendency to be roughly constant on much larger scales.

The authors of [43] have performed some tests in order to investigate the linear scaling range using the radial structure function $s(r)$. In Fig. 7, $\log s(r)$ is plotted for the sample S65. Open circles and asterisks represent different procedure for the edge correction: the standard method [22] and a new method offered by Coleman at al(1988). As one can see, the $s(r)$ does not differ too much from the results of the calculation done using the standard method. The power-law behaviour with $\alpha = 1.16$(dotted line) can be appreciated for r less than $5h^{-1}$ Mpc, it violates for larger values and tends to be 1 beyond $15h^{-1}$ Mpc. Such behaviour gave to V.J.Martinez and B.J.T. Jones the reason to title their article "Why the Universe is not a fractal".

Having no intention to go to the discussion, we note that the transition from one regime to the other can be described by the formula (3.7), (9.4)

$$s(r) = 1 + \xi(r) \sim 1 + (r/r_0)^{\alpha-3},$$

where $\alpha = 1.16$ and $r_0 = 5.6$Mpc (asterisks are omitted).

The resulting function coincides with observation data (Fig.7).

14. Conclusion

The following conclusions can be made from what has been considered in this paper.

I. The offered mesofractal model describing correlated point structures is characterized by four parameters:the characteristic exponent α, the surviving probability c, the mean density of initial points n_0 and a space scale parameter.

II. The model satisfactorily reproduce the main characteristics of the visible distribution of galaxies in the Universe including the relation between two-point and three-point correlation functions.

III. The model yields the cell count distribution in the form of gamma (or negative binomial) distribution that is in accordance with a known representation.

IV. On small scales, the model reveals fractal properties of the galaxy distribution, on large scales, it corresponds to the homogeneous Poissonian distribution (see Fig.8, where the Monte Carlo results of two-dimensional version of the model are demonstrated).

V. In the transition domain, its prediction coincide with observation data.

Nevertheless, a few questions remain without answers. Perhaps, the most important of them : how correspond averaging over Gibbs' realizations and averaging over different domains in the same realization? In other words, where the ergodic hypotheses is valid, on what scales. To avoid misunderstanding, we stress in conclusion again: the offered model *has no relation to dynamics of the Universe*, it is only one of the possible ways to represent available information in condensed form.

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